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FACULTY OF APPLIED SCIENCE AND ENGINEERING
SCHOOL OF ENGINEERING RESEARCH

BULLETIN NO. 6

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The Application of Certain Projections to the Construction of General Maps of Canada

By

L. B. Stewart

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The Application of Certain Projections to the Construction of General Maps of Canada

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THE APPLICATION OF CERTAIN PROJECTIONS TO THE CONSTRUCTION OF GENERAL MAPS OF CANADA

By LOUIS B. STEWART, *Professor of Surveying and Geodesy*

For the representation of a considerable portion of the surface of the earth numerous methods have been devised and used in the construction of maps, and among the best of these are to be included the different modifications of the conical projection with two standard parallels. It is here proposed to examine some of these modes of representation with a view to determining their adaptability to maps of the Dominion of Canada.

In all the modifications of the projection to be examined the parallels of latitude are represented by arcs of circles having a common centre, and the meridians by straight lines radiating from the same centre. Also, the scale is true along the two standard parallels, situated near the northern and southern limits of the map. The scale in longitude is therefore slightly reduced between the standard parallels, and enlarged outside of them. Various additional assumptions may be made, giving maps suited for special purposes. Thus, it may be assumed that areas shall be correctly shown; or, that all small portions shall be shown in their true forms.

An obvious advantage of these projections is that they may be easily constructed. If the scale of the map is not too large the parallels may be drawn by the aid of beam compasses; and, the degrees of longitude having been laid off along the standard parallels, the meridians may then be drawn by joining the points of division by straight lines. If the scale is large the standard parallels may be constructed by means of the rectangular coordinates of their points of intersection with the meridians, the axes of coordinates being the central meridian and lines at right angles to it through the points of intersection of the parallels.

Other advantages are: the meridians and parallels intersect at right angles; the scale is the same at all points on a parallel, either in latitude or longitude, and therefore the scale error, or the distortion, does not increase with increase of longitude from the centre of the map, as is the case with some projections that have been extensively used. These projections are therefore well adapted for maps covering a wide range in longitude, and at the same time having a considerable extent in latitude.

For the sake of comparison brief descriptions of the Polyconic and Bonne's projections will be given at the end of this account.

The following modifications of the type projection will be considered in turn:

(1) The scale in latitude is to be true at all points, and the standard parallels are to be selected arbitrarily.

(2) The scale in latitude is to be true at all points, and the maximum scale error in longitude between the standard parallels is to be equal, with opposite sign, to that of either extreme parallel.

(3) The area of any portion of the map is to be correctly shown. This projection is known as Albers' Equivalent, or Equal-Area, Projection.

(4) All small portions of the map shall be shown in their true forms. This gives Lambert's Conformal Projection.

In order to institute a comparison between these projections the following quantities will be determined in each case:

The scale errors in latitude and longitude;

The error in the representation of areas;

The error in the representation of directions at a point.

The following notation will be used:

ϕ will denote the latitude of a parallel;

ϕ_1 and ϕ_2 those of the southern and northern standard parallels;

ϕ' and ϕ'' those of the extreme parallels;

ω the longitude of a meridian; reckoned from the central meridian of the map;

θ the angle at the centre of a developed parallel corresponding to a given value of ω;

r the radius of a developed parallel;

n the ratio $\theta : \omega$, known as the "constant of the cone";

ρ the radius of curvature of the meridian at a point in given latitude;

N the length of the normal at a point, terminated in the axis of rotation of the terrestrial spheroid;

P the radius of a parallel of latitude;

m the length of a given meridian arc.

Convenient expressions for ρ N and P are as follows:

$$\rho = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi)^{\frac{3}{2}}} \tag{1}$$

$$N = \frac{a}{(1-e^2 \sin^2 \phi)^{\frac{1}{2}}} \tag{2}$$

$$P = N \cos \phi \tag{3}$$

in which a denotes the radius of the equator, and e the eccentricity of the elliptic meridian. Substituting

$$\sin \psi = e \sin \phi \tag{4}$$

these become

$$\rho = a(1-e^2) \sec^3 \psi \tag{5}$$

$$N = a \sec \psi \tag{6}$$

$$P = a \sec \psi \cos \phi \tag{7}$$

For the Clarke spheroid of 1866:

$$\begin{aligned} &\log a \text{ (in miles)} = 3.5980536 \\ &\log e \qquad\qquad\quad = \bar{2}.9152513 \\ &\log (1-e^2) \qquad = \bar{1}.9970504 \end{aligned}$$

The length of a meridian arc may be found by means of the expression:

$$\begin{aligned} m = &[1.83919449]\Delta\phi \text{ (in degrees)} \\ &-[1.3043545] \cos 2\phi_0 \sin \Delta\phi \\ &+[\bar{2}.3301509] \cos 4\phi_0 \sin 2\Delta\phi \\ &-[\bar{5}.45093] \cos 6\phi_0 \sin 3\Delta\phi + \end{aligned} \tag{8}$$

in which—

$\Delta\phi$ denotes the difference of latitude of the extremities of the arc, and

ϕ_0 the mean of the extreme latitudes.

The numbers in rectangular brackets are the logarithms of numerical coefficients.

For a short meridian arc—less than 1°—the following expression may be used:

$$m = \rho . \Delta\phi \sin 1'' \tag{9}$$

$\Delta\phi$ being the amplitude of the arc in seconds, and ρ the radius of curvature at its middle point.

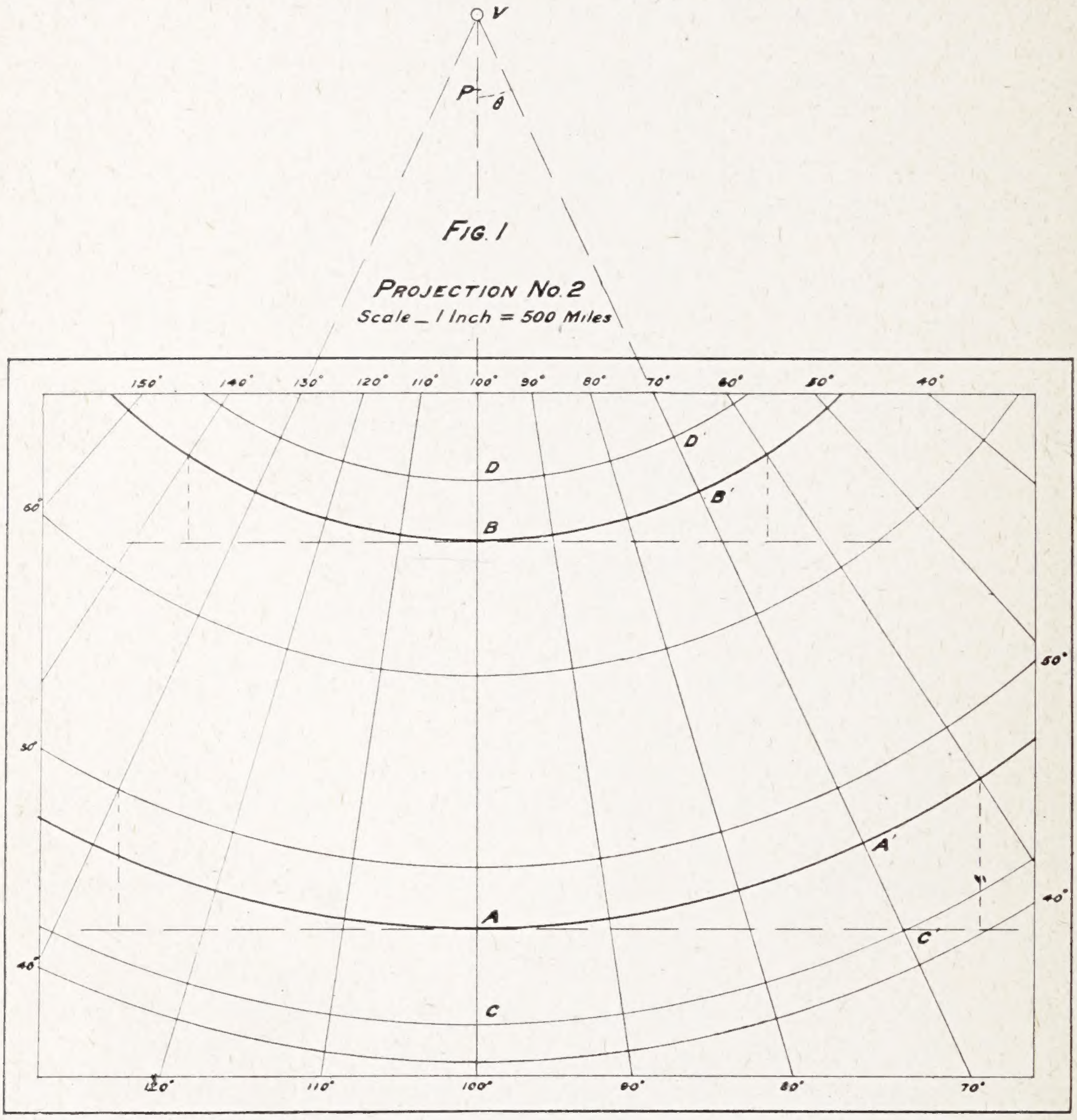
The Supplement to the Manual of Instructions for the Survey of Dominion Lands contains a table giving, for every 10′ of latitude from 42° to 70°, the values of the logarithms of $\rho \sin 1''$ (ρ being there denoted by R) $N \sin 1''$ and $P \sin 1''$, the unit being the Gunter's chain. The log. of either of these quantities in miles may be found by adding to the tabular quantity

$$\log \frac{1}{80 \sin 1''} = 3.4113351.$$

The table on p. 11 below, giving the lengths of degrees of latitude from 40° to 70°, may be used for finding the length of a long arc, the fractional degrees being found by eq. (9).

We shall proceed to the investigation of the projections enumerated above.

(1) The scale in latitude is true at all points, and the standard parallels are selected arbitrarily.



In Fig. 1 AA' and BB' are the standard parallels, their radii being r_1 and r_2, P_1 and P_2 denoting their true radii. Then we have

$$\frac{VA}{VB}=\frac{AA'}{BB'}=\frac{N_1 \cos \phi_1 . \omega}{N_2 \cos \phi_2 . \omega}=\frac{P_1}{P_2}$$

$$\therefore \quad \frac{VA}{VA-VB}=\frac{P_1}{P_1-P_2}$$

or

$$\frac{r_1}{m}=\frac{P_1}{P_1-P_2}$$

or

$$r=\frac{P_1 m}{P_1-P_2} \tag{10}$$

m denoting the length of the meridian arc AB. Also

$$r_2 = r_1 - m \tag{11}$$

Similarly, the radius of any developed parallel is

$$r = r_1 \pm m' \tag{12}$$

m' being the length of the corresponding meridian arc.

Again,

$$AA' = r_1\theta = P_1\omega$$

so that

$$n = \frac{\theta}{\omega} = \frac{P_1}{r_1} = \frac{P_1 - P_2}{m} \tag{13}$$

Also, denoting the scale ratios in latitude and longitude by σ and σ', respectively, we have

$$\sigma = 1$$

$$\sigma' = \frac{r\theta}{P\omega} = n\frac{r}{P} = \frac{P_1 - P_2}{m} \cdot \frac{r}{P} \tag{14}$$

Also the area scale is given by the expression

$$\frac{rdrd\theta}{\rho d\phi \, . \, Pd\omega}$$

But $dr = \rho d\phi$, and $d\theta = nd\omega$, $\therefore$ this expression becomes

$$\frac{rn}{P} = \sigma';$$

so that σ' is also the area scale, which is otherwise evident.

To investigate the local misrepresentation of direction at a point let us consider an infinitesimal portion of a line drawn upon the surface of the earth and having the azimuth α, and assume that a meridian is drawn through one extremity, and a line perpendicular to the meridian through the other extremity, thus forming a small triangle, the lengths of whose sides are $\rho d\phi$ and $Pd\omega$, respectively; so that we have

$$\tan \alpha = \frac{Pd\omega.}{\rho d\phi}$$

Then, if α' denote the angle on the map corresponding to α, we have

$$\tan \alpha' = \frac{\sigma' Pd\omega}{\sigma \rho d\phi} = \frac{\sigma'}{\sigma} \tan \alpha \tag{15}$$

An expression for $\alpha' - \alpha$ may be obtained in the form of a series. Eq. (15) is of the form

$$\tan x = p \tan y,$$

which by expansion in series gives

$$x - y = q \sin 2y + \tfrac{1}{2}q^2 \sin 4y + \tfrac{1}{3}q^3 \sin 6y$$

in which

$$q = \frac{p-1}{p+1}$$

Applying this, we have

$$\alpha'-\alpha=\frac{\sigma'-\sigma}{\sigma'+\sigma}\sin 2\alpha+\tfrac{1}{2}\left(\frac{\sigma'-\sigma}{\sigma'+\sigma}\right)^2\sin 4\alpha+$$

If $\alpha=45°$ the first term of this series has its maximum value, the second term vanishes, and the third—as well as each succeeding term that does not vanish—is extremely small, so that the maximum value o $\alpha'-\alpha$ in seconds is, very approximately:

$$\alpha'-\alpha=\frac{\sigma'-\sigma}{(\sigma'+\sigma)\sin 1''} \tag{16}$$

In the present case this becomes

$$\alpha'-\alpha=\frac{\sigma'-1}{(\sigma'+1)\sin 1''} \tag{17}$$

In order to determine the latitude, between those of the standard parallels, for which the scale error in longitude is a maximum we have

$$\frac{d\sigma'}{d\phi}=0,\text{ or } n\frac{d}{d\phi}\left(\frac{r}{P}\right)=0.$$

But
$$\frac{d}{d\phi}\left(\frac{r}{P}\right)=\frac{P\frac{dr}{d\phi}-r\frac{d\phi}{dP}}{P^2};$$

∴ the condition for a maximum is:

$$P\frac{dr}{d\phi}=r\frac{dP}{d\phi}$$

Then, as
$$r=r_1-m$$
$$\therefore\quad \frac{dr}{d\phi}=-\frac{dm}{d\phi}=-\rho.$$

Also
$$P=a\cos\phi\,(1-e^2\sin^2\phi)^{-\frac{1}{2}}$$
$$\begin{aligned}\therefore\quad \frac{dP}{d\phi}&=ae^2\sin\phi\cos^2\phi(1-e^2\sin^2\phi)^{-\frac{3}{2}}\\&\quad-a\sin\phi\,(1-e^2\sin^2\phi)^{-\frac{1}{2}}\\&=\frac{ae^2\sin\phi\cos^2\phi-a\sin\phi(1-e^2\sin^2\phi)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}\\&=-\frac{a(1-e^2)\sin\phi}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}\\&=-\rho\sin\phi\end{aligned}$$

∴ the condition for a maximum becomes

$$\rho N\cos\phi=(r_1-m)\rho\sin\phi$$

or
$$\tan\phi=\frac{N}{r_1-m}=\frac{N}{r}. \tag{18}$$

This equation must be solved by trials.

The coordinates of the intersections of the meridians and parallel may be computed by means of the equations:

$$\theta = n\omega \quad x = r \sin\theta \qquad (19)$$
$$y = r(1-\cos\theta)$$

The following table gives the length in miles of 1° of latitude for different latitudes.

Lat.	Length of 1°	Lat.	Length of 1°
°	m	°	m
40	68.99917	55	69.18039
41	69.01131	56	69.19181
42	69.02351	57	69.20306
43	69.03575	58	69.21414
44	69.04801	59	69.22502
45	69.06029	60	69.23570
46	69.07256	61	69.24616
47	69.08482	62	69.25639
48	69.09704	63	69.26637
49	69.10921	64	69.27609
50	69.12132	65	69.28555
51	69.13335	66	69.29471
52	69.14529	67	69.30361
53	69.15711	68	69.31219
54	69.16882	69	69.32046
55		70	

We shall now apply the above projection to the construction of a map of the Dominion, by assuming

$$\phi' = 42° \quad \phi'' = 70°$$

and the extreme longitudes to be 55° and 141° west. The latitudes of the standard parallels are also assumed to be:

$$\phi_1 = 45°, \ \phi_2 = 60°.$$

By eq. (10), and the above table, we find

$$r_1 = 3548.3829\text{m}$$
$$r_2 = 2511.2187\text{m}$$

Also, by eq. (13)

$$n = 0.791123.$$

Also, by eq. (14)—

$$\sigma' = 1.00722, \text{ for lat. } 42°$$
$$\sigma' = 1.05811, \text{ for lat. } 70°.$$

These are also the area scales at the extreme latitudes.

The latitude of the parallel, between the two standard parallels, at which the scale ratio in longitude is a maximum is, eq. (18):

$$\phi = 52° \ 56' \ 05''$$

and for this latitude

$$\sigma' = 0.9914455$$

For this latitude we also find—eq. (17):

$$\alpha'-\alpha = = -14'\ 46''.04$$

At latitude 63° we find

$$\sigma' = 1.010086$$

so that up to latitude 63° the scale error will barely exceed one per cent.

By assuming the latitudes of the standard parallels differently a different distribution of scale error may be obtained.

(2) The maximum scale error between the standard parallels is to be equal, with opposite sign, to that of the extreme parallels.

Let z denote the length VP, P being the projection of the pole, and m' and m'' the lengths of the meridian arcs between the extreme parallels and the pole.

Then, the scale ratios of these parallels being equal, we have

$$\frac{(z+m')\theta}{P'\omega} = \frac{(z+m'')\theta}{P''\omega};$$

whence

$$z = \frac{m'P''-m''P'}{P'-P''}; \tag{20}$$

which determines z.

Again, to find the latitude corresponding to the maximum scale error between the standard parallels, we have, for a parallel near the centre of the map

$$1-\sigma' = 1-\frac{n(z+m)}{P},$$

or

$$\sigma' = \frac{n(z+m)}{P}. \tag{21}$$

We now differentiate this with reference to ϕ and equate to zero, thus

$$\frac{d\sigma'}{d\phi} = n\frac{P\dfrac{dm}{d\phi}-(z+m)\dfrac{dP}{d\phi}}{P^2} = 0$$

$$\therefore \qquad P\frac{dm}{d\phi} = (z+m)\frac{dP}{d\phi}.$$

Then as

$$\frac{dm}{d\phi} = -\rho \text{ and } \frac{dP}{d\phi} = -\rho \sin\phi$$

this becomes

$$P\rho = (z+m)\rho\sin\phi,$$

or

$$N\cos\phi = (z+m)\sin\phi,$$

or

$$N_0\cot\phi_0 - m_0 = z, \tag{22}$$

which gives the latitude corresponding to the maximum scale error This equation must be solved by trials.

Again, the scale ratio being the same, with opposite sign, on this parallel and either extreme parallel, we have

$$1-\frac{n(z+m_0)}{P_0}=\frac{n(z+m')}{P'}-1$$

whence

$$\frac{1}{n}=\frac{z+m_0}{2P_0}+\frac{z+m'}{2P'} \tag{23}$$

which gives n.

Again, for the standard parallels

$$\frac{n(z+m_1)}{P_1}=\frac{n(z+m_2)}{P_2}=1$$

or

$$P_1-nm_1=nz, \tag{24}$$

and

$$P_2-nm_2=nz;$$

by which ϕ_1 and ϕ_2 may be determined by trials.

This modification of (1) may now be applied to the construction of a map of Canada by assuming again the extreme latitudes to be

$$\phi'=42^\circ \quad \phi''=70^\circ$$

We must first find m' and m'' By means of eq. (8) we find—

$$m''=1387^{m}.5858$$

and by adding to this the lengths of degrees of latitude contained in the table, p. 11 we find— $m'=3324^{m}.6545$.

Then by table in D.L.S. Manual—

$$\log P'=3.4697861,$$
$$\log P''=3.1334070.$$

Then eq. (20) gives

$$z=268.5985.$$

Then by eq. (22)—

$$\phi_0=57^\circ\ 49'\ 20''.17.$$

The value of n then follows by (23):

$$n=0.8334627\ldots;$$

for which the scale ratio is found by (21):

$$\sigma'=0.9847151$$
$$=1-0.0152849$$

As a check the scale ratio is found for either extreme parallel to be

$$\sigma'=1.0152848$$

The maximum error in a direction for latitude ϕ_0 is found by (17) to be

$$\alpha'-\alpha=-26'\ 28''.5;$$

and this is the maximum for the whole map.

The latitudes of the standard parallels are, by (24)

$$\phi_1=46^\circ\ 53'\ 39''.15$$
$$\phi_2=66^\circ\ 53'\ 34''.81.$$

(3) Albers' Equivalent, or Equal-Area, Projection.

A brief development of the theory of this projection may not be out of place.

The scale ratios in latitude and longitude may be thus expressed

$$\sigma = -\frac{dr}{\rho d\phi}, \quad \sigma' = \frac{rd\theta}{Pd\omega} = \frac{nr}{N\cos\phi} \tag{25}$$

Also the condition for equivalence of areas is:

$$\sigma\sigma' = 1$$

or

$$-\frac{dr}{\rho d\phi} \cdot \frac{nr}{N\cos\phi} = 1$$

or

$$nrdr = -\rho N\cos\phi d\phi \tag{26}$$

By (1) and (2) this becomes

$$nrdr = -\frac{a^2(1-e^2)}{(1-e^2\sin^2\phi)^2}\cos\phi d\phi$$

or

$$rdr = -\frac{a^2(1-e^2)}{n} \cdot \frac{\cos\phi d\phi}{(1-e^2\sin^2\phi)^2}$$

Therefore

$$\frac{r^2}{2} = -\frac{a^2(1-e^2)}{n}\int\frac{\cos\phi d\phi}{(1-e^2\sin^2\phi)^2}$$

If now r_0 denote the value of r when $\phi = 0$ this becomes

$$r^2 - r_0^2 = -\frac{2a^2(1-e^2)}{n}\int_0^\phi \frac{\cos\ \phi d\phi}{(1-e^2\sin^2\phi)^2}$$

If then the earth were a sphere having the radius R, and β denote the value of the latitude on this supposition, the right-hand member of the last expression would become:

$$\frac{2R^2}{n}\int_0^\beta \cos\beta d\beta = -\frac{2R^2}{n}\sin\beta.$$

We may therefore define β by equating the two expressions, which gives

$$R^2\sin\beta = a^2(1-e^2)\int_0^\phi \frac{\cos\phi d\phi}{(1-e^2\sin^2\phi)^2}$$

Therefore, expanding and integrating each term separately, we have

$$R^2\sin\beta = a^2(1-e^2)\int_0^\phi (1+2e^2\sin^2\phi+3e^4\sin^4\phi$$
$$+4e^6\sin^6\phi+)\cos\phi d\phi$$
$$= a^2(1-e^2)\ (\sin\phi+\frac{2e^2}{3}\sin^3\phi+\frac{3e^4}{5}\sin^5\phi$$
$$+\frac{4e^6}{7}\sin^7\phi+).$$

R may be determined by assuming that when $\phi = \frac{\pi}{2}$, β also $= \frac{\pi}{2}$, which

gives

$$R^2=a^2(1-e^2)\left(1+\frac{2e^2}{3}+\frac{3e^4}{5}+\frac{4e^6}{7}+\right) \tag{27}$$

R is the radius of the sphere having the same surface area as that of the spheroid. For the Clarke spheroid of 1866—

$$\log R(\text{in metres})=6.80420742$$
$$\log R(\text{in miles})=3.59756243$$

β is then given by the expression:

$$\sin\beta=\sin\phi\frac{1+\frac{2e^2}{3}\sin^2\phi+\frac{3e^4}{5}\sin^4\phi+\frac{4e^6}{7}\sin^6\phi+}{1+\frac{2e^2}{3}+\frac{3e^4}{5}+\frac{4e^6}{7}+}$$

$$=\sin\phi\left\{1-\frac{2e^2}{3}\cos^2\phi-\left(\frac{7}{45}\cos^2\phi+\frac{3}{5}\sin^2\phi\cos^2\phi\right)e^4\right.$$
$$-\left(\frac{64}{945}\cos^2\phi+\frac{6}{35}\sin^2\phi\cos^2\phi+\frac{4}{7}\sin^4\phi\cos^2\phi\right)e^6$$
$$\left.+\text{ etc.}\right\} \tag{28}$$

β is termed the "authalic", or equal-area, latitude. The difference $\phi-\beta$ may be computed by the expression:

$$\phi-\beta=467''.0129\sin 2\phi-0''.4494\sin 4\phi$$
$$+0''.0005\sin 6\phi— \tag{29}$$

A table is given in Special Publication No. 67 of the U.S. Coast and Geodetic Survey which contains values of $\phi-\beta$ for $30'$ intervals of latitude from $0°$ to $90°$,

The expression for r may then be written:

$$r^2=r_0^2-\frac{2R^2}{n}\sin\beta \tag{30}$$

The constants n and r_0 must now be determined. Introducing the condition that the scale shall be true along the standard parallels, we have

$$r_1=\frac{N_1\cos\phi_1}{n}\quad r_2=\frac{N_2\cos\phi_2}{n} \tag{31}$$

and substituting these values of r_1 and r_2 in the expression for r we have

$$r_0^2-\frac{2R^2}{n}\sin\beta_1=\frac{N_1^2\cos^2\phi_1}{n^2}=\frac{P_1^2}{n^2}$$

$$r_0^2-\frac{2R^2}{n}\sin\beta_2=\frac{N_2^2\cos^2\phi_2}{n^2}=\frac{P_2^2}{n^2}$$

Then subtracting we find

$$\frac{2R^2}{n}\left(\sin\beta_2-\sin\beta_1\right)=\frac{P_1^2-P_2^2}{n^2}$$

whence

$$n=\frac{P_1^2-P_2^2}{2R^2\,(\sin\beta_2-\sin\beta_1)} \tag{32}$$

Again, from the general expression for r we have

$$r_1^2=r_0^2-\frac{2R^2}{n}\sin\beta_1$$

whence

$$r^2-r_1^2=\frac{2R^2}{n}\;(\sin\beta_1-\sin\beta)$$

or

$$r^2=r_1^2+\frac{2R^2}{n}(\sin\beta_1-\sin\beta). \tag{33}$$

Similarly

$$r^2=r_2^2+\frac{2R^2}{n}(\sin\beta_2-\sin\beta) \tag{34}$$

For convenience of computation the expressions for r may be written

$$\frac{r^2}{R^2}=\left(\frac{r_1^2}{R^2}+\frac{2}{n}\sin\beta_1\right)-\frac{2}{n}\sin\beta \tag{35}$$

$$\frac{r^2}{R^2}=\left(\frac{r_2^2}{R^2}+\frac{2}{n}\sin\beta_2\right)-\frac{2}{n}\sin\beta \tag{36}$$

The scale ratio for any parallel is

$$\sigma'=\frac{nr}{P}, \tag{37}$$

and that for a meridian at any point:

$$\sigma=\frac{1}{\sigma'}=\frac{P}{nr} \tag{38}$$

To find the latitude of the parallel, between the standard parallels, for which σ is a maximum, we have

$$\frac{d\sigma}{d\phi}=\frac{1}{n}\,\frac{r\dfrac{d}{d\phi}(N\cos\phi)-N\cos\phi\,\dfrac{dr}{d\phi}}{r^2};$$

so that, equating to zero, the condition for a maximum is

$$r\frac{d}{d\phi}(N\cos\phi)=N\cos\phi\,\frac{dr}{d\phi}.$$

But $\dfrac{d}{d\phi}(N\cos\phi)=\cos\phi\,\dfrac{dN}{d\phi}-N\sin\phi$

$$= -\cos\phi \frac{a}{2}\ (1-e^2\sin^2\phi)^{-\frac{3}{2}}(-2e^2\sin\phi\cos\phi)$$

$$-N\sin\phi$$

$$= \frac{ae^2\sin\phi\cos^2\phi - a\sin\phi\ (1-e^2\sin{}^2\phi)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}$$

$$= -\frac{a(1-e^2)\sin\phi}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} = -\rho\sin\phi$$

Also, by (26) $$\frac{dr}{d\phi} = -\frac{\rho N\cos\phi}{nr},$$

Therefore the condition for a maximum becomes

$$r\rho\sin\phi = \frac{N\cos\phi\rho N\cos\phi}{nr}$$

$$= \frac{\rho N^2\cos^2\phi}{nr}$$

or $$\tan\phi\sec\phi = \frac{N^2}{nr^2} \qquad (39)$$

By eq. (16) we have for the maximum error in direction at a point

$$\alpha'-\alpha = \frac{\frac{1}{\sigma}-\sigma}{\left(\frac{1}{\sigma}+\sigma\right)\sin\ 1''} = \frac{1-\sigma^2}{(1+\sigma^2)\sin 1''} \qquad (40)$$

We now apply this projection to the construction of a map of the Dominion, by assuming as before

$$\phi' = 42^\circ \quad \phi'' = 70^\circ$$

We shall also take the standard parallels in the latitudes

$$\phi_1 = 45^\circ \quad \phi_2 = 65^\circ$$

By eq. (32) we find

$$n = 0.8067798\ldots$$

Also by eq's, (31)

$$r_1 = 3479.5229$$
$$r_2 = 2081.8885$$

The latitude for which σ is a maximum is found by a series of approximations, using eq. (39), to be:

$$56^\circ\ 17'\ 05''.04;$$

for which we find

$$\sigma = 1.01539135\ldots$$

Also $$\sigma' = 0.9848418\ldots$$
$$= 1 - 0.0151582.\ldots$$

We also find—eq. (40)—for the same latitude:

$$\alpha' - \alpha = -52' \ 30''.28.$$

The following table gives the value of r for each degree of the map computed by eq. (36), from which are found the length of each degree of the developed meridian. The errors of the latter are found by comparison with the lengths given in the table, p. 11.

Lat.	r	Length of 1°	Error	
°	m	m	m	%
42	3685.8212	68.5672	−0.4563	−0.66
43	3617.2540	68.7695	−0.2662	−0.39
44	3548.4845	68.9616	−0.0864	−0.13
45	3479.5229	69.1430	0.0827	0.12
46	3410.3799	69.3137	0.2411	0.35
47	3341.0362	69.4725	0.3877	0.53
48	3271.5937	69.6193	0.5223	0.76
49	3201.9744	69.7530	0.6438	0.93
50	3132.2214	69.8734	0.7521	1.09
51	3032.3480	69.9789	0.8453	1.22
52	2992.3691	70.0393	0.9243	1.34
53	2922.2995	70.1438	0.9867	1.43
54	2852.1557	70.2009	1.0312	1.49
55	2781.9557	70.2380	1.0576	1.53
56	2711.7177	70.2552	1.0634	1.54
57	2641.4625	70.2513	1.0482	1.51
58	2571.2112	70.2226	1.0085	1.46
59	2500.9886	70.1684	0.9434	1.36
60	2430.8202	70.0831	0.8504	1.23
61	2360.7341	69.9725	0.7263	1.05
62	2290.7616	69.8244	0.5680	0.82
63	2220.9372	69.6384	0.3720	0.54
64	2151.2988	69.4103	0.1342	0.19
65	2081.8885	69.1349	−0.1506	−0.22
66	2012.7536	68.8062	−0.4885	−0.70
67	1943.9474	68.4185	−0.8851	−1.28
68	1875.5289	67.9625	−1.3497	−1.95
69	1807.5664	67.4307	−1.8898	−2.73
70	1740.1357			

(4) The Lambert Conformal Projection.

A brief development of the theory of this projection is also given here.

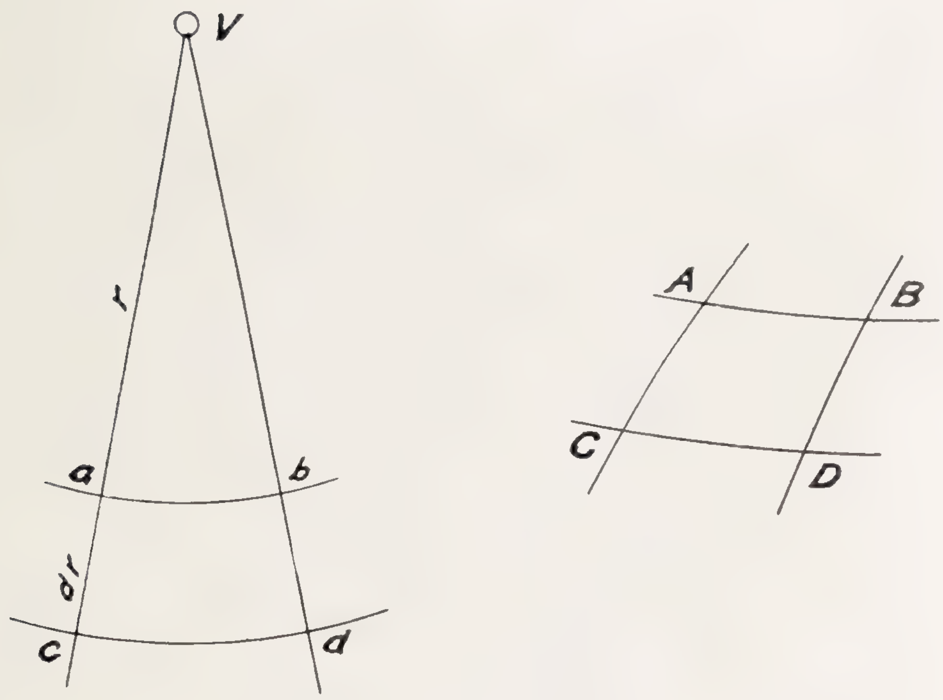


FIG. 2

In Fig. 2 $ABCD$ represents an elementary quadrilateral on the surface of the earth, bounded by meridians and parallels of latitude, and $abcd$ is its representation on the map.

If u denotes the co-latitude of AB we have

$$BD = \rho du = \frac{a(1-e^2)\,du}{(1-e^2\cos^2 u)^{\frac{3}{2}}}$$

$$AB = N \sin u d\omega = \frac{a \sin u d\omega}{(1-e^2\cos^2 u)^{\frac{1}{2}}}$$

Then, as the representation is conformal

$$\frac{bd}{ab} = \frac{BD}{AB}$$

or

$$\frac{dr}{nr d\omega} = \frac{a(1-e^2)du}{(1-e^2\cos^2 u)^{\frac{3}{2}}} \cdot \frac{(1-e^2\cos^2 u)^{\frac{1}{2}}}{a \sin u d\omega}$$

$$\therefore \quad \frac{1}{n}\frac{dr}{r} = \frac{(1-e^2)du}{\sin u(1-e^2\cos^2 u)}$$

$$= \frac{1}{\sin u}\left(1-e^2\frac{1-\cos^2 u}{1-e^2\cos^2 u}\right)du$$

$$= \frac{du}{\sin u} - \frac{e^2 \sin u du}{1-e^2\cos^2 u}. \tag{41}$$

Then, resolving into partial fractions, this becomes

$$\frac{1}{n}\frac{dr}{r} = \frac{du}{\sin u} - \frac{e^2 \sin u du}{2(1+e\cos u)} - \frac{e^2 \sin u du}{2(1-e\cos u)}.$$

Then integrating, we have

$$\frac{1}{n}\log r=\log\tan\frac{u}{2}+\frac{e}{2}\log(1+e\cos u)$$

$$-\frac{e}{2}\log(1-e\cos u)+\frac{1}{n}\log k$$

the last term being the constant of integration. This may be written

$$\log\frac{r}{k}=n\log\left\{\tan\frac{u}{2}\left(\frac{1+e\cos u}{1-e\cos u}\right)^{\frac{e}{2}}\right\}$$

or $$r=k\left(\tan\frac{u}{2}\right)^n\left(\frac{1+e\cos u}{1-e\cos u}\right)^{\frac{ne}{2}}$$

If now an angle z be assumed, such that

$$\tan\frac{z}{2}=\tan\frac{u}{2}\left(\frac{1+e\cos u}{1-e\cos u)}\right)^{\frac{e}{2}};$$

and also an angle q, such that

$$\cos q=e\cos u \tag{42}$$

we shall have

$$\left(\frac{1+e\cos u}{1-e\cos u}\right)^{\frac{1}{2}}=\left(\frac{1+\cos q}{1-\cos q}\right)^{\frac{1}{2}}=\cot\tfrac{1}{2}q;$$

so that $$\left(\cot\frac{q}{2}\right)^e=\left(\frac{1+e\cos u}{1-e\cos u}\right)^{\frac{e}{2}}$$

and $$\tan\frac{z}{2}=\tan\frac{u}{2}\left(\cot\frac{q}{2}\right)^e; \tag{43}$$

$\therefore$ $$r=k\left(\tan\frac{z}{2}\right)^n \tag{44}$$

A close approximation to the angle z may be found by computing the geocentric latitude ϕ' by the equation:

$$\tan\phi'=(1-e^2)\tan\phi; \tag{45}$$

then approximately

$$z=90°-\phi' \tag{46}$$

The value of n may now be found by assuming the scale to be true along the two selected standard parallels. Then making use of the fact that the lengths of two arcs of these parallels, each having the longitude ω, are proportional to their representations, we have

$$\frac{n\omega r_1}{n\omega r_2}=\frac{N_1\cos\phi_1.\omega}{N_2\cos\phi_2.\omega}$$

or $$\frac{r_1}{r_2}=\frac{N_1\cos\phi_1}{N_2\cos\phi_2}$$

or $$\left(\frac{\tan \frac{1}{2}z_1}{\tan \frac{1}{2}z_2}\right)^n = \frac{N_1 \cos \phi_1}{N_2 \cos \phi_2} = \frac{P_1}{P_2}.$$

Then taking log's. and solving for n, we have

$$n = \frac{\log P_1 - \log P_2}{\log \tan \frac{z_1}{2} - \log \tan \frac{z_2}{2}} \tag{47}$$

An expression for k may now be found by equating the length of an arc of a standard parallel and that of its representation, thus

$$nr_1\omega = P_1\omega$$

or $$nk\left(\tan \frac{z_1}{2}\right)^n = P_1$$

or $$k = \frac{P_1}{n\left(\tan \frac{z_1}{2}\right)^n} = \frac{P_2}{n\left(\tan \frac{z_2}{2}\right)^n} \tag{48}$$

The collected equations then are:

$$\cos q = e \cos u = e \sin \phi$$

$$\tan \frac{z}{2} = \tan \frac{u}{2}\left(\cot \frac{q}{2}\right)^e = \tan (45° - \tfrac{1}{2}\phi) \left(\cot \frac{q}{2}\right)^e$$

$$n = \frac{\log P_1 - \log P_2}{\log \tan \frac{z_1}{2} - \log \tan \frac{z_2}{2}}$$

$$k = \frac{P_1}{n\left(\tan \frac{z_1}{2}\right)^n} = \frac{P_2}{n\left(\tan \frac{z_2}{2}\right)^n}$$

$$r = k\left(\tan \frac{z}{2}\right)^n$$

Eq's. (19) may be used in plotting a parallel of latitude by coordinates.

The scale ratio of any parallel is evidently:

$$\sigma' = \frac{n\omega r}{P\omega} = \frac{nr}{P}; \tag{49}$$

see eq. (14). Also, as the projection is conformal

$$\sigma = \sigma' \tag{50}$$

The area scale is

$$\sigma\sigma' = \sigma^2 = \left(\frac{nr}{P}\right)^2; \tag{51}$$

Eq, (16) also gives $\alpha' - \alpha = 0$; (52)

so that directions are correctly represented at a point.

To find the latitude for which the scale error, between the standard parallels, is a maximum, we have, as on p. 10 the necessary condition:

$$P\frac{dr}{d\phi} = r\frac{dP}{d\phi}$$

Then, by (41)

$$\frac{dr}{d\phi} = nr\left(\frac{e^2 \cos\phi}{1-e^2 \sin^2\phi} - \frac{1}{\cos\phi}\right)$$

Also on p. 10 it was shown that

$$\frac{dP}{d\phi} = -\rho \sin\phi.$$

Therefore the above condition becomes

$$Pn\left(\frac{e^2 \cos\phi}{1-e^2 \sin^2\phi} - \frac{1}{\cos\phi}\right) = -\rho \sin\phi$$

or $$n\,\frac{a\cos\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} \cdot \frac{e^2\cos^2\phi - 1 + e^2\sin^2\phi}{\cos\phi(1-e^2\sin^2\phi)} = -\rho\sin\phi$$

or $$-n\,\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} = -\rho\sin\phi$$

or $$n\rho = \rho\sin\phi$$

or $$\sin\phi = n, \tag{53}$$

the required condition.

Applying this projection to the same data as in the previous case, viz.:

$$\phi' = 42°,\ \phi'' = 70°$$

$$\phi_1 = 45°,\ \phi_2 = 65°$$

we obtain the following results:

Eq. (47)— $n = 0.8234698\ldots$

Eq. (31)— $r_1 = 3409.0005$

$r_2 = 2039.6931.$

Also the latitude for which the scale error is a maximum is—eq. (53):

$$\phi = \sin^{-1} n = 55°\ 26'\ 01''.16;$$

and for this latitude we find—eq. (49):

$$\sigma = \sigma' = 0.984733\ldots$$
$$= 1 - 0.015\ldots$$

Also the area scale—eq. (51):

$$\sigma^2 = 0.969699$$
$$= 1 - 0.030.\ldots$$

Directions are correctly represented—eq. (52).

The properties of projections (3) and (4) are evidently mutually

exclusive, as it is impossible for any projection to embody, for all parts of the map, the two conditions:

$$\sigma\sigma' = 1, \text{ and } \sigma = \sigma',$$

unless, at the same time, it were possible that—

$$\sigma = 1, \text{ and } \sigma' = 1.$$

This could only be realized, however, if the surface of a sphere or a spheroid were developable.

The following table gives, for Lambert's projection, the value of r for each degree of latitude between 42° and 70°, from which the lengths of the individual degrees of latitude follow. The errors of the latter are found by comparison with the lengths given in the table, p. 11.

Lat.	r	Length of 1°	Error	
°	m	m	m	%
42	3617.0780	69.5744	0.5509	0.80
43	3547.5036	69.3532	0.3175	0.46
44	3478.1504	69.1499	0.1019	0.15
45	3409.0005	68.9644	−0.0959	−0.14
46	3340.0361	68.7972	−0.2754	−0.40
47	3271.2389	68.6478	−0.4370	−0.63
48	3202.5911	68.5166	−0.5804	−0.84
49	3134.0745	68.4040	−0.7052	−1.02
50	3065.6705	68.3096	−0.8117	−1.17
51	2997.3609	68.2341	−0.8992	−1.30
52	2929.1268	68.1775	−0.9678	−1.40
53	2860.9493	68.1402	−1.0169	−1.47
54	2792.8091	68.1227	−1.0461	−1.51
55	2724.6864	68.1251	−1.0553	−1.53
56	2656.5613	68.1482	−1.0436	−1.51
57	2588.4131	68.1924	−1.0107	−1.46
58	2520.2207	68.2586	−0.9555	−1.38
59	2451.9621	68.3474	−0.8776	−1.27
60	2383.6147	68.4593	−0.7764	−1.12
61	2315.1554	68.5956	−0.6506	−0.94
62	2246.5598	68.7576	−0.4988	−0.72
63	2177.8022	68.9463	−0.3201	−0.46
64	2108.8559	69.1628	−0.1133	−0.16
65	2039.6931	69.4090	0.1235	0.18
66	1970.2841	69.6870	0.3923	0.57
67	1900.5971	69.9986	0.6950	1.00
68	1830.5985	70.3463	1.0341	1.49
69	1760.2522	70.7327	1.4122	2.04
70	1689.5195			

THE POLYCONIC PROJECTION

In this projection each parallel of latitude is assumed to be the base of a right circular cone, tangent to the surface of the earth along the

parallel, and whose vertex is consequently on the axis of the earth produced. When the cone is developed the parallel becomes a circular arc whose radius—

$$r = N \cot \phi, \qquad (54)$$

The central meridian of the map is represented by a straight line, on which the true lengths of the degrees of latitude are laid off to scale. Each parallel is then described about its own centre, and the degrees of longitude laid off on it correctly to scale. Through the points of division thus found the remaining meridians are drawn,

As the scale is true along any parallel we have the relation:

$$r\theta = P\omega;$$

or

$$\theta N \cot \phi = \omega N \cos \phi;$$

so that

$$\theta = \omega \sin \phi; \qquad (55)$$

and

$$n = \frac{\theta}{\omega} = \sin \phi. \qquad (56)$$

n has therefore a different value for each parallel.

The parallels may be drawn, on maps of large scale, by means of coordinates computed by (19).

The scale of the map is thus correct along the central meridian, and along the parallels, but the scale in latitude increases considerably with increase of distance from the central meridian. The meridians, in general, are represented by curved lines, and they do not intersect the parallels at right angles.

The increase of scale in latitude is given approximately by the expression (See U.S.C. and G. Survey Spec. Pub. No, 57):

$$E = 0.01\left(\frac{\omega \cos \phi}{8.1}\right)^2 \qquad (57)$$

in which ω is expressed in degrees. By means of this equation we find the scale error at some points as follows:

$\phi = 49°$, $\angle = 125°$: $E = 0.0410$ or 4.10%.
$\phi = 55°\ 26'$, $\angle = 130°$: $E = 0.0442$ or 4.42%.
$\phi = 70°$, $\angle = 141°$: $E = 0.02997$ or 2.997%.

Let us now find, for latitude 49°, the range of longitude for which the scale error in latitude amounts to 0.01. Placing $E = 0.01$ in (57) and solving for ω we find

$$\omega = 8.1 \sec \phi$$
$$= 12°.346\ldots$$

for latitude 49°. This projection should therefore not be used for maps covering a wider range in longitude than about 25°. It is, however, within these limits, a very useful projection, on account of the ease with which it can be constructed; extensive tables being available which

are applicable to any part of the world, and which greatly facilitate the work of the cartographer.

BONNE'S PROJECTION

This is a modification of the simple conical projection. The scale is true along a central parallel, and the degrees of latitude are laid off correctly along the central meridian which is represented by a straight line. The radius of the central parallel:

$$r_0 = N_0 \cot \phi_0, \tag{58}$$

ϕ_0 being its latitude, and the remaining parallels are circular arcs described about the same centre, so that the radius of any parallel:

$$r = r_0 \pm m, \tag{59}$$

m being its distance, measured along the central meridian, from the central parallel. The degrees of longitude are then laid off truly along each parallel and the points of division joined, as in the polyconic projection. We have then, for any parallel, the relation

$$r\theta = N \cos \phi . \omega$$

whence

$$\theta = \frac{N}{r} \omega \cos \phi \tag{60}$$

The points of intersection of the meridians and parallels may be plotted by means of coordinates, computed by the equations:

$$x = r \sin \theta$$
$$y = r_0 - r \cos \theta \tag{61}$$

the origin being the point of intersection of the central meridian and the central parallel.

A point in favour of this projection is that it is equal-area; the main objection to it is on account of the obliquity of the intersections of the meridians and parallels at points distant from the central meridian. The divergence from a right angle at any such intersection may be found by the expression

$$\tan \psi = \frac{\pi}{180} (\theta - \omega \sin \phi), \tag{62}$$

θ and ω being expressed in degrees. For latitude 70° and longitude 141° ($\omega = 41°$) this gives $\psi = 5° \; 54' \; 20''$.

Also the scale error in latitude at this point is

0.005 or 0.5 per cent.

At the point whose geographical coordinates are

$$\phi = 49°, \; L = 125°$$

the scale error in latitude is zero, and

$$\psi = 1° \; 43'09''.6.$$

The scale errors in latitude thus appear to be small. This advantage, however, is offset by the fact that the obliquity of the intersections of the meridians and parallels causes the two diagonals of a quadrilateral formed by two meridians and two parallels, which should have the same length, to have widely different lengths. This fault does not exist in the four modifications of the conical projection with two standard parallels, considered above, in which the intersections are all right-angled.